Fourier Analysis

Note Title 2/20/2020

Review: Let $f \in R[-\pi, \pi]$.

$f * P_r(x) \longrightarrow f(x)$ if f is cts at x.

If f is cts everywhere, the limit is unif.

§2.5. Application to the heat equation on the unit disc.

Let $D = \{ (x,y) \in \mathbb{R}^2 : x^2+y^2 < 1 \}$

In polar coordinates (r, θ),

$$D = \{ (r, \theta) : 0 \le r < 1 \}.$$

Let $f \in R[-\pi, \pi]$. Define

$$u = u(r,\theta) = f * P_r(\theta), \quad (r,\theta) \in D.$$

Thm (1) $u \in C^2(D)$ and $\Delta u = \frac{\partial^2}{\partial r^2} u + \frac{\partial u}{r \partial r} + \frac{\partial^2 u}{r^2 \partial \theta^2} = 0$.

(2) If f is cts at θ, then

$$\lim_{r \to 1} u(r, \theta) = f(\theta)$$

If f is cts everywhere, the limit is unif.

(3) If f is cts on the circle, then
$u = u(r,\theta)$ satisfies $\Delta u = 0$

Moreover, u is the unique solution of $\Delta u = 0$ satisfying both ① and ②.

Pf. (i) Notice that

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} \hat{f}(n) e^{in\theta}.$$

$$|\hat{f}(n)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|\,dx, \quad \forall\, n \in \mathbb{Z}$$

Moreover, for any $0<\rho<1$, the series

$$\sum_n r^{|n|} \hat{f}(n) e^{in\theta},$$

$$\sum_n \frac{\partial}{\partial r}\left(r^{|n|} \hat{f}(n) e^{in\theta} \right)$$

$$\sum_n \frac{\partial}{\partial \theta}\left(r^{|n|} \hat{f}(n) e^{in\theta} \right)$$

converge unif on $\{ (r,\theta) : 0 \leq r < \rho \}$.

So u is diff on D. (Indeed u is infinite diff on D)

(ii) If θ is a continuity pt of f, then

$$u(r,\theta) \to f(\theta) \text{ as } r \to 1$$

which is an appucation of the convergence Thm.

(iii)

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{r\,\partial r} + \frac{\partial^2 u}{r^2 \partial \theta^2}$$

$$= \sum_{n=-\infty}^{\infty} \Delta\left(r^{|n|} \hat{f}(n)\, e^{in\theta} \right)$$

$$= 0$$

$$\left(\text{e.g. } \Delta\left(r^3 e^{i3\theta} \right) = 6r \cdot e^{i3\theta} + 3re^{i3\theta} + (i3)^2 \cdot re^{i3\theta} = 0 \right)$$

To prove the uniqueness result, let

$v = v(r,\theta)$ be another solution of $\Delta v = 0$ satisfying ① and ②.

For a fixed $0 < r < 1$, write

$$v(r,\theta) \sim \sum_{n=-\infty}^{\infty} a_n(r)\, e^{in\theta}.$$

where
$$a_n(r) := \frac{1}{2\pi} \int_{-\pi}^{\pi} v(r,\theta)\, e^{-in\theta}\, d\theta$$

Recall that
$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{r^2 \partial \theta^2} = 0$$

Let $n \in \mathbb{Z}$. Taking integration gives

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{\partial^2 v}{r^2\partial\theta^2}\right)e^{-in\theta}\,d\theta$$

$$\Rightarrow \quad a_n(r)''+\frac{1}{r}a_n(r)'+\underbrace{\frac{(ni)^2}{r^2}\cdot a_n(r)}_{:=\,-\frac{n^2}{r^2}a_n(r)}=0$$

However, the general solution of the above ODE is

$$a_n(r)=\begin{cases} A\,r^{|n|}+B\,r^{-|n|}, & n\in\mathbb{Z}\setminus\{0\} \\ A+B\log r, & n=0 \end{cases}$$

Notice $a_n(r)$ is bdd in $\{0<r<1\}$, hence $B=0$.

Hence

$$v=v(r,\theta)\sim\sum_{n=-\infty}^{\infty}A_n r^{|n|}\cdot e^{in\theta}.$$

As $v(r,\cdot)$ is C^2,

we have $\quad v(r,\theta)=\sum_{n=-\infty}^{\infty}A_n r^{|n|}e^{in\theta}.$

Notice $v(r,\theta)\rightrightarrows f(\theta)$ as $r\to 1$.

So for any given $n\in\mathbb{Z}$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}v(r,\theta)e^{-in\theta}\,d\theta\to\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\theta)e^{-in\theta}\,d\theta$$

as $r\to 1$.

That is, $A_n r^{|n|} \to \hat{f}(n)$ as $r \to 1$

Hence $A_n = \hat{f}(n)$.

Therefore,

$$v(r,\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{in\theta}$$

$$= u(r,\theta) \quad \blacksquare.$$

Chap 3. Convergence of Fourier Series.

§3.1 Recall: If f is cts on the circle so that

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty,$$

then

$$S_N f(x) \rightrightarrows f(x) \quad \text{on the Circle.}$$

In this chapter, we present some more general results on the convergence of Fourier Series.

① Mean square convergence.

<u>Thm 1</u>: Let $f \in \mathcal{R}[-\pi,\pi]$, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - S_N f(x)|^2 \, dx \to 0 \text{ as } N \to \infty.$$

(L^2- convergence)

② Pointwise convergence.

Thm2. Let $f \in \mathcal{R}[-\pi, \pi]$.

Assume that f is diff. at x_0.

Then

$$S_N f(x_0) \to f(x_0) \quad \text{as } N \to \infty$$

③ Examples of continuous functions on the circle with divergent Fourier series.

§3.2. Inner product spaces.

Def. Let V be a vector space on $\mathbb{C}$

An inner product on V over $\mathbb{C}$ is a map

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C} \quad \text{so that}$$

(1) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (conjugate symmetry)

(2) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$, $\forall\ \alpha, \beta \in \mathbb{C}$

(3) $\langle x, x \rangle \geq 0$.

Def. $\|x\| = \sqrt{\langle x, x \rangle}$, $\forall x \in V$.

Thm: Let V be an inner product space over $\mathbb{C}$.

① (Pythagorean Thm)

If $\langle x, y \rangle = 0$, then

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

② (Cauchy-Schwartz inequality)

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|.$$

③ (triangle inequality)

$$\|x+y\| \leq \|x\| + \|y\|.$$

Pf. (1) Assume $\langle x, y \rangle = 0$.

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y \rangle \\ &= \|x\|^2 + \|y\|^2 + \langle x, y \rangle + \langle y, x \rangle \\ &= \|x\|^2 + \|y\|^2 \end{aligned}$$

(2) Let x, y in V.

Let $r = |\langle x, y \rangle|$.

WLOG, assume that $r > 0$, otherwise we have nothing to prove.

Then $\langle x, y\rangle = r e^{i\theta}$ for some $\theta \in [0, 2\pi)$.

Let $t \in \mathbb{R}$, define

$$f(t) = \| x + t e^{i\theta} y \|^2$$

$$= \langle x + t e^{i\theta} y,\ x + t e^{i\theta} y \rangle$$

$$= \|x\|^2 + t^2 \|y\|^2 + \langle x, t e^{i\theta} y \rangle + \langle t e^{i\theta} y, x \rangle$$

$$= \|x\|^2 + t^2 \|y\|^2 + 2rt.$$

Hence f is a quadratic poly taking non-negative values.

It follows that

$$(2r)^2 \leq 4 \cdot \|x\|^2 \|y\|^2.$$

Equivalently

$$r \leq \|x\| \|y\|.$$

(3)

$$\|x + y\|^2 = \langle x+y, x+y \rangle$$

$$= \|x\|^2 + \|y\|^2 + \langle x, y \rangle + \langle y, x \rangle$$

$$\leq \|x\|^2 + \|y\|^2 + 2\|x\| \cdot \|y\|$$

(using the Cauchy-Schwarz)

$$= (\|x\| + \|y\|)^2.$$

So $\|x+y\| \leq \|x\| + \|y\|$. ▨